Assignment 10

Problems 1–3 are for systems of differential equations.

1. Consider the Cauchy problem for the third order equation:

$$y''' = f(x, y, y'),\ y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_3.$$

   Express it as the Cauchy problem for a system of three first order equations.

2. Optional. Consider (2.3) for a system of differential equations. Let $f \in C(G)$ satisfies the Lipschitz condition on every compact subset of the open set G. Show that there exists a solution y^* of (2.3) defined on some (α, β) with the properties: (a) whenever y is a solution of (2.3) on some I, $I \subset (\alpha, \beta)$; and (b) when β is finite, for each compact K in G, there exists some $\rho > 0$ such that $(x, y^*(x)) \in G \setminus K$ for $x \in [\beta - \rho, \beta)$.

3. Consider the Cauchy problem (2.3) for a system of differential equations. Assuming that $f(x, y) \in C(\mathbb{R}^2)$ satisfying the Lipschitz condition and the growth condition

$$\big|f(x, y)\big| \le C(1 + |y|), \quad \forall (x, y) \in \mathbb{R}^2.$$

   Show that (2.3) admits a global solution, that is, a solution defined on $(-\infty, \infty)$. Hint: Study the differential inequality satisfied by $|y|^2 = \sum_{j=1}^n y_j^2$.

4. Let $I_n = [-n, n]$ and $\|f\|_n = \sup\{|f(x)| : x \in [-n, n]\}$. For $f, g \in C(\mathbb{R})$, define

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}.$$

   (a) Show that this is a complete metric on $C(\mathbb{R})$. Hint: Use Problem 8, Ex 4.

   (b) Show that $\{f_j\}$ converges to f in this metric implies $\{f_j\}$ converges to f uniformly on every bounded interval.

5. Optional. Inspired by the previous problem, introduce a metric on $C^\infty(\mathbb{R})$, the vector space of all smooth functions on the real line, so that $\{f_n\}$ converges to f in this metric implies $\{d^k f_n/dx^k\}$ converges to $d^k f/dx^k$ uniformly on each bounded interval for each $k \ge 0$.

6. Let E be a bounded, convex set in $\mathbb{R}^n$. Show that a family of equicontinuous functions is bounded in E if it is bounded at a single point, that is, if there is some constant M such that $|f(x_0)| \le M$ for all f in this family.

7. Let $\{f_n\}$ be a sequence in $C(G)$ where G is open in $\mathbb{R}^n$. Suppose that on every compact subset of G, it is equicontinuous and bounded. Show that there is a subsequence $\{f_{n_j}\}$ converging to some $f \in C(G)$ uniformly on each compact subset of G.

8. Let $\{f_n\}$ be a sequence of bounded functions in $[0, 1]$ and let F_n be

$$F_n(x) = \int_0^x f_n(t)dt.$$

   (a) Show that the sequence $\{F_n\}$ has the Bolzano-Weierstrass property provided there is some M such that $\|f_n\|_\infty \le M$, for all n.

(b) Show that the conclusion in (a) holds when boundedness is replaced by the weaker condition: There is some K such that

$$\int_0^1 |f_n|^2 \leq K, \quad \forall n.$$

9. Let $K \in C([a,b] \times [a,b])$ and define the operator T by

$$(Tf)(x) = \int_a^b K(x,y)f(y)dy.$$

   (a) Show that T maps $C[a,b]$ to itself.

   (b) Show that whenever $\{f_n\}$ is a bounded sequence in $C[a,b]$, $\{Tf_n\}$ contains a convergent subsequence.

10. Let f be a bounded, uniformly continuous function on $\mathbb{R}$. Let $f_a(x) = f(x+a)$. Show that for each $l > 0$, there exists a sequence of intervals $I_n = [a_n, a_n + l]$, $a_n \to \infty$, such that $\{f_{a_n}\}$ converges uniformly on $[0, l]$.

11. Optional. Let $\{h_n\}$ be a sequence of analytic functions in the unit disc satisfying $|h_n(z)| \leq M$, $\forall z, |z| < 1$. Show that there exist an analytic function h in the unit disc and a subsequence $\{h_{n_j}\}$ which converges to h uniformly on each smaller disc $\{z : |z| \leq r\}$, $r \in (0,1)$. Suggestion: Use a suitable Cauchy integral formula.